Talk 1st february: The geometric template (When does a filtration point towards the closure of an orbit?)

Recall:

- Definition: A good moduli space for an algebraic stack $\mathcal{X}$ is a map $q : \mathcal{X} \to Y$ where Y is an algebraic space st $q_* : QCoh(\mathcal{X}) \to QCoh(Y)$ is exact and the canonical map is an equivalence $\mathcal{O}_Y \simeq q_* \mathcal{O}_{\mathcal{X}}$.

  The basic example of a good moduli space morphism is the GIT quotient map:
  $$[Z/G] \longrightarrow Z /\!/ G.$$
  where G is a linearly reductive group acting over Z projective scheme.

- Theorem (mumford): Let $G = GL_n(\mathbb{C})$ act algebraically on a variety Z then:
  - The closure of an orbit contains a unique closed orbit
  - The map $\pi : Z \longrightarrow Z/\!/G$ is surjective
  - $x, y \in Z$ have the same image applying π iff $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$

- Recall that if Z admits an ample linearisation L of the action of G then we say that:
  - A point $x \in Z$ is (semi)stable if for all $\lambda : \mathbb{C}^* \to G$ 1-PS, $\mu(x, \lambda)(\leq) 0$ where $\mu(x, \lambda) = -wt(L|_0)$
  - A point $x \in Z$ is unstable if $\exists \lambda$ 1-PS st $\mu(x, \lambda) > 0$.

Moreover the points in GIT quotient correspond to the closed orbits in the set of semistable points and that every point in $\overline{Orb}(x) \cap Z^{ss}$ with $x \in Z^{ss}$ can be obtained as $\lim_{z \to \infty} \lambda(z) \cdot x$ with $\mu(\lambda, x) = 0$ (J-H - filtration which "maximizes" the numerical invariant)
↳ it is already bounded by 0

The geometric template.

Let us work through the classic example, let $X = [Z/G]$ we have the morphism $[Z/G] \longrightarrow BG = [*/G]$ that is relatively representable by projective schemes. If Y is a regular 2-dim noetherian scheme and $(0,0) \in Y$ is a closed point then any morphism $Y \setminus \{(0,0)\} \to BG$ extends uniquely to a morphism $p : Y \to BG$. The morphism p does not necessarily lift to X but the lift $\xi : Y \setminus \{(0,0)\} \to X$ defines a section of the morphism $Y \times_{BG} X \to Y$ over $Y \setminus \{(0,0)\}$. Let Σ be the closure of the image of this section, thus we have the following diagram

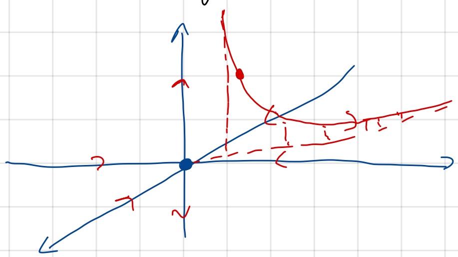
$$\begin{tikzcd}
Y \setminus \{(0,0)\} \arrow[r, dashed] \arrow[rr, hook, bend left] \arrow[d, "\xi"'] & \Sigma \arrow[r, dashed] \arrow[dl, "\bar{\xi}"] & Y \arrow[d, dashed, "p"] \\
X \arrow[rr] & & BG
\end{tikzcd}$$

Note that: i) if Y has a G_m-action fixing $(0,0)$ and ξ is G_m equivariant, then the dotted arrows can be filled G_m-equivariantly

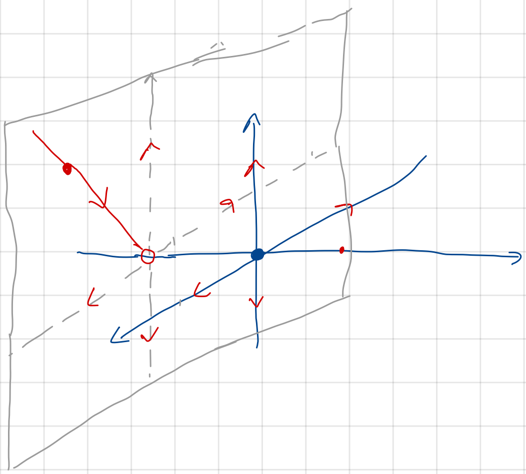
ii) if we have a linearization of the G action on Z, by a line bundle L, then $\bar{\xi}^* L$ will be ample on Σ if L is relatively ample for the map $X \to BG$

This implies that for any point $x \in |X|$ and a given filtration $\Theta \to X$ st $\Theta(1) = x$ we can determine if x is a closed orbit by applying the Hilbert-Mumford criterion to $\bar{\xi}^* L$ (i.e (semi) stability is determined in this manner). Moreover, if a point is not semistable (i.e there is $f : \Theta \to X$ st $\mathrm{wt}\, f^* L > 0$) then by Harder Narasimhan theory there is a unique filtration that maximizes the weight.

And for every $x \in |X|$ there is a "local quotient presentation" $f : [A_k / G_m] \to X$ (see Existence of moduli spaces for algebraic stacks, Def 2.2) where f is a non-degenerate filtration.



- if x is a stable point of the action $G \curvearrowright Z$ then any filtration $f : \Theta \to \mathcal{X}$ with $f(1) = x$ satisfies $wg(f^*L) < 0$, thus the orbit $G \cdot x$ is closed (in particular the 1-PS determined by f is also closed). Finally if $\pi : \mathcal{X} \to |\mathcal{X}|$ then x can be considered as a point of $|\mathcal{X}|$ which classifies its orbit.



- if x is a semistable point then any filtration $f : \Theta \to \mathcal{X}$ with $f(1) = x$ satisfies $wg(f^*L) \leq 0$, by HN boundedness there is $f_0 : \Theta \to \mathcal{X}$ st $wg(f_0^*L) = 0$ thus the orbit of x is not closed. Thus the point in $\Sigma / Y \setminus (0,0)$ $\hat{x}$ is precisely the closure of the 1-PS determined by f. Finally if $\pi : \mathcal{X} \to |\mathcal{X}|$, then the point $\hat{x}$ can be considered as a closed point of $|\mathcal{X}|$ and f identifies it with x. $\hat{x}$ classifies the closure of the orbit $G \cdot x$.

- If x is unstable then the HN property and the number $wg(f^*L)$ determine a stratification of the points of $|\mathcal{X}|$.

In conclusion we have that the map $\pi : \mathcal{X} \to |\mathcal{X}|$ restricted to $\pi : [Z^{ss}/G] \to [Z^{ss}/G]/\sim$ where $\sim$ is the relation which identifies points to the closure of the HN 1-PS is a good moduli space.

Now we apply the same methods to $\mathrm{Coh}^d(X)$ generalizing carefully.

recall: Monotonicity via "∞-dim GIT" gave us the diagram:

$$\begin{tikzcd}
Y\setminus(0,0) \arrow[dr, "\tau"] \arrow[ddr, bend right, "j"'] \arrow[drr, bend left, "b"] & & \\
& Gr_M \arrow[r] \arrow[d] & M \arrow[d] \\
& Y \arrow[r, "g"] & M_{rat}
\end{tikzcd}$$

where we distinguished $\Sigma \subset Gr_M$ as the closure of the image of τ

Recall also that the polynomial invariant ν defined from the line bundles L_n over M is strictly S-monotone so Σ has a G_m action fixing $(0,0)$ and the morphism $\bar{\varepsilon}: \Sigma \longrightarrow M$ is G_m-equivariant, thus the pullbacks $\bar{\varepsilon}^*(L_n)$ will be ample on Σ for all $n >> 0$ this will allow us to not only identify which points of M are (semi) stable, ν already tells us, but thanks to the HN property of ν it will give us the closure of the orbits and ultimately the relation $\sim$ on points.

- If $x \in |M|$ is a stable point of M then by definition $M^{\mu}(x) < 0$ this means that for any filtration $f: \Theta \longrightarrow M$ st $f(1) = x$. By Rees's construction we can identify f with a filtration of x that we will note by $E_\bullet$, we know that $\nu(f) := \dfrac{wt(L_n|_0)}{\sqrt{b(f|_0)}} = \dfrac{\sum_{m\in\mathbb{Z}} m\left(\bar{p}_{E_m/E_{m+1}} - \bar{p}_{E}\right)\cdot rk_{E_m/E_{m+1}}}{\sqrt{\sum_{m\in\mathbb{Z}} rk_{E_m/E_{m+1}}\cdot m^2}}$

  Thus we recovered the Gieseker stable pure sheaves on X.

  Since there are no destabilizing filtrations we don't identify x with no other point

- If $x \in |\mathcal{M}|$ is a semistable point then by definition $M^{\nu}(x) \leq 0$ and by HN boundedness there is a filtration $f: \Theta \to \mathcal{M}$ with $\nu(f) = 0$. Again by Rees's construction this corresponds to a filtration $\mathcal{F}_\bullet$ of x we thus have the diagram:

$$\begin{tikzcd} \mathcal{E} \subset \mathrm{Gr}_{\mathrm{Coh}}^{\leq N, P} \arrow[r] \arrow[d] & \mathrm{Coh} \\ Y|(0,0) \arrow[r, hook] \arrow[ur] & Y \end{tikzcd}$$

Where the point corresponding to $\mathcal{E}|_{(0,0)}$ is $\mathrm{gr}(\mathcal{F}_\bullet)$ by uniqueness of the $\mathbb{G}_m$-filling property.

Thus we have $\nu(f) = \sum_{m \in \mathbb{Z}} m \left(\bar{p}_{\mathcal{F}_m / \mathcal{F}_{m+1}} - \bar{p}_{\mathcal{F}}\right) \cdot \mathrm{rk}_{\mathcal{F}_m / \mathcal{F}_{m+1}} = 0$, $\bar{p}_{\mathcal{F}_m / \mathcal{F}_{m+1}} = \bar{p}_{\mathcal{F}}$ and by construction each $\mathcal{F}_m / \mathcal{F}_{m+1}$ is stable, this is the Jordan Hölder filtration. Thus the stack $[\mathcal{E}/\mathbb{G}_m] \to *$ with a good moduli space identifies $\mathrm{gr}\,\mathcal{F}_\bullet$ with $\mathcal{F}$ in $|\mathrm{Coh}|$ by proposition 4.4 of Existence of Moduli Spaces for algebraic Stacks.

Evenmore we have that $\mathrm{Coh}^{ss} \to |\mathrm{Coh}^{ss}|/\sim$ is a good moduli space by the same proposition.

Remark: Theorem 4.1 ([AHLH]) builds the good moduli space by covering Coh with quotient presentation around every $x \in |\mathcal{X}|$ i.e

$f: ([\mathrm{Spec}\, A / GL_N], 0) \to (\mathcal{X}, x)$ which is an étale and affine pointed morphism.

This can also be done for $\mathrm{Coh}(X)$ by taking neighborhoods around every point $x \in |\mathrm{Coh}|$.

↳ Constructed by using "Perturbation of filtrations" (see Theorem 3.60 HL 18) which gives a bijection between filtrations of a point $x \in \mathcal{X}$ which are "close" to a filtration f and filtrations of the graded obj which are "close" to a canonical filt of $f(0)$.

4.2. **Proof of the existence result.** We first provide conditions on an algebraic stack ensuring that there are local quotient presentations which are Θ-surjective and stabilizer preserving. This is the key ingredient in the proof of Theorem 4.1.

Proposition 4.4. *Let $\mathcal{Y}$ be an algebraic stack, locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space S, and let $y \in |\mathcal{Y}|$ be a closed point. Let $f\colon (\mathcal{X}, x) \to (\mathcal{Y}, y)$ be a pointed étale and affine morphism such that there exists an adequate moduli space $\pi\colon \mathcal{X} \to X$ and f induces an isomorphism $f|_{f^{-1}(\mathcal{G}_y)}$ over the residual gerbe at y (e.g. f is a local quotient presentation).*

(1) If $\mathcal{Y}$ is Θ-reductive, then there exists an affine open subspace $U \subset X$ of $\pi(x)$ such that $f|_{\pi^{-1}(U)}$ is Θ-surjective.
(2) If $\mathcal{Y}$ has unpunctured inertia, then there exists an affine open subspace $U \subset X$ of $\pi(x)$ such that $f|_{\pi^{-1}(U)}$ which induces an isomorphism $I_{\pi^{-1}(U)} \to \pi^{-1}(U) \times_{\mathcal{Y}} I_{\mathcal{Y}}$.

In particular, if $\mathcal{Y}$ is locally linearly reductive, is Θ-reductive and has unpunctured inertia, then there exists a local quotient presentation $g\colon \mathcal{W} \to \mathcal{Y}$ around y which is Θ-surjective and induces an isomorphism $I_{\mathcal{W}} \to \mathcal{W} \times_{\mathcal{Y}} I_{\mathcal{Y}}$.

We can then glue these local quotient presentations by lemma 4.5.

Lemma 4.5. *Let $\mathcal{X}$ be a locally noetherian algebraic stack with affine diagonal. Suppose that $\{\mathcal{U}_i\}_{i\in I}$ is a Zariski-cover of $\mathcal{X}$ such that each $\mathcal{U}_i$ admits a good moduli space and each inclusion $\mathcal{U}_i \hookrightarrow \mathcal{X}$ is Θ-surjective. Then $\mathcal{X}$ admits a good moduli space.*

With this construction we recover the moduli of Gieseker semistable pure sheaves on X with Hilbert polynomial P. We can recover the properness of the moduli space by checking that $\mathrm{Coh}^d(X)$ satisfies the existence part of the valuative criterion for properness. And we can recover the stratification by Harder Narasimhan by applying the same methods to unstable points.

remark: * We never used a global action to construct $\mathrm{Coh}^{\nu\text{-}ss}(X)$.

* The related moduli problems $\mathrm{Pair}^d(X)$ and $\Lambda\mathrm{Coh}^d(X)$ have rational stacks so we can repeat the entire construction on them, since their numerical invariant comes from pullback through the forget morphism $\mathcal{M} \xrightarrow{\text{for}} \mathrm{Coh}$.